$\mathbb{RP}^n$-like Lagrangians in $\mathbb{CP}^n$

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(Joint work with M. Konstantinov)

Big goal : Classify Lagrangian submanifolds $L \subset \mathbb{CP}^n$

subject to natural topological restrictions

$\rightsquigarrow$

L looks like $\mathbb{RP}^n$

Inspiration

X smooth Fano variety of complex dimension n

Defn: Minimal Chern number

$$N_X := \text{non-negative generator of } \langle c_1(X), \pi_2(X) \rangle$$

Thm (Mori–Mukai conjecture, Cho–Miyaoka–Shepherd-Barron '02):

$$N_X \leq n+1 \quad \text{with equality iff} \quad X \cong \mathbb{CP}^n$$

Question: Is there a Lagrangian analogue of this?

Fano variety $\rightsquigarrow$ monotone Lagrangian $L \subset X = \mathbb{CP}^n$

rules out small Lagrangians in Darboux charts

N_X $\rightsquigarrow$ minimal Maslov number

$N_L :=$ generator of $\langle \mu_L, \pi_2(X,L) \rangle$

$H^2(X,L;\mathbb{Z}) \ni$ Maslov class

Try to classify monotone Lagrangians of maximal N_L

Reminder on Maslov class

(X^{2n}, ω) symplectic, $L^n \subset X$ Lagrangian

$(\Lambda^n_{\mathbb{C}} TX, \Lambda^n_{\mathbb{R}} TL)$ classified by $\varphi : (X, L) \longrightarrow (BU(1), BO(1)) \simeq (\mathbb{CP}^\infty, \mathbb{RP}^\infty)$

$$\begin{array}{ccccccccc}
\cdots \to & H^1(\mathbb{RP}^\infty) & \to & H^2(\mathbb{CP}^\infty, \mathbb{RP}^\infty) & \to & H^2(\mathbb{CP}^\infty) & \to & H^2(\mathbb{RP}^\infty) & \to \cdots \\
& \| & & \wr\| & & \wr\| & & \wr\| & \\
& 0 & & \mathbb{Z}\cdot m & \xhookrightarrow{m \mapsto 2h} & \mathbb{Z}\cdot h & \twoheadrightarrow & \mathbb{Z}/2 &
\end{array}$$

$c_1(X) := \varphi^*(h) \in H^2(X; \mathbb{Z})$ $\qquad$ $\mu_L := \varphi^*(m) \in H^2(X, L; \mathbb{Z})$

L is <u>monotone</u> if $\exists \lambda > 0$ st $\langle \omega, A \rangle = \lambda \langle \mu_L, A \rangle$ for all $A \in \pi_2(X, L)$.

Note: $\mu_L|_X = 2c_1(X)$ so $N_L \mid 2N_X$

Monotone Lagrangians in $\mathbb{CP}^n$

$\mathbb{RP}^n \subset \mathbb{CP}^n$ is monotone with $N_L = n+1$ $\Leftarrow 2H_1(L;\mathbb{Z}) = 0$

Thm (Seidel '00): All monotone $L \subset \mathbb{CP}^n$ have $N_L \leq n+1$.

Always have $N_L \mid 2N_X = 2(n+1)$

No known equality cases except $\mathbb{RP}^n$.

Question (Biran - Cornea '07): Is every monotone $L \subset \mathbb{CP}^n$ with $N_L = n+1$ Hamiltonian isotopic to $\mathbb{RP}^n$?

Sketch proof that $N_L \leq n+1$

Know $N_L \mid 2N_X = 2(n+1)$ so suppose $N_L = 2(n+1)$

Have Oh spectral sequence $E_1 = H^*(L, \Lambda_{\mathbb{Z}/2}) \Rightarrow HF^*(L, L; \Lambda_{\mathbb{Z}/2})$

$\Lambda_{\mathbb{Z}/2} = \mathbb{Z}/2[T^{\pm 1}]$ Novikov ring, $\deg T = N_L$

$$E_1 = \ldots \to T^{-1} \cdot H^{*+N_L}(L; \mathbb{Z}/2) \xrightarrow{\deg 1} H^*(L; \mathbb{Z}/2) \xrightarrow{\deg 1} T \cdot H^{*-N_L}(L; \mathbb{Z}/2) \xrightarrow{\deg 1} T^2 \cdot H^{*-2N_L}(L; \mathbb{Z}/2) \to \ldots$$

All differentials vanish for degree reasons.

eg $H^i(L; \mathbb{Z}/2) \longrightarrow T \cdot H^{i+1-(2n+2)}(L; \mathbb{Z}/2)$

Deduce $HF^*(L, L; \Lambda_{\mathbb{Z}/2}) \cong H^*(L; \Lambda_{\mathbb{Z}/2})$ as $\Lambda_{\mathbb{Z}/2}$-modules

__Sketch__ proof that $N_L \leq n+1$

Have $HF^*(L,L;\Lambda_{\mathbb{Z}/2}) \cong H^*(L;\Lambda_{\mathbb{Z}/2})$ as $\Lambda_{\mathbb{Z}/2}$-modules

$\Lambda_{\mathbb{Z}/2} = \mathbb{Z}/2[T^{\pm 1}]$ Novikov ring, $\deg T = N_L$

Have path γ in $\widetilde{\mathrm{Ham}}(\mathbb{CP}^n)$ from $\mathrm{id}_{\mathbb{CP}^n}$ to grading shift by 2

$$\gamma(t)\cdot[z_0 : z_1 : \ldots : z_n] = [e^{2\pi i t} z_0 : z_1 : \ldots : z_n]$$

So $L \cong L[2]$ and hence $HF^* \cong HF^{*+2}$

But $HF^n \cong H^n \cong \mathbb{Z}/2$

$HF^{n+2} \cong H^{n+2} = 0$

$\#$

Previous results $\quad L \subset \mathbb{CP}^n$ monotone, $\quad N_L = n+1$

Biran, '06
building on Seidel '00
Biran-Cieliebak '01

$$H^*(L;\mathbb{Z}/2) \cong H^*(\mathbb{RP}^n;\mathbb{Z}/2) \text{ additively}$$

Biran-Cornea, '07, '09

$$H^*(L;\mathbb{Z}/2) \cong H^*(\mathbb{RP}^n;\mathbb{Z}/2) \text{ as rings}$$

Damian, '12

$$\begin{cases} n \text{ odd} & \tilde{L} \underset{\text{homeo}}{\cong} S^n, \quad \pi_1(L) \text{ finite} \\ n \text{ even} & \tilde{L} \ \mathbb{Z}/2\text{-HS}, \quad \pi_1(L) \cong \mathbb{Z}/2 \end{cases}$$

Borman-Li-Wu, '14 $\quad$ for $n=2$ L is Ham isotopic to $\mathbb{RP}^2$

Main tools

- Oh spectral sequence
- Closed-open string map $\quad CO^0 : QH^*(\mathbb{CP}^n) \longrightarrow HF^*(L,L)$

$$\Lambda[h]/(h^{n+1}-T^2) \cong QH^*(\mathbb{CP}^n)$$

Quantum version of restriction

$$H^*(\mathbb{CP}^n) \longrightarrow H^*(L)$$

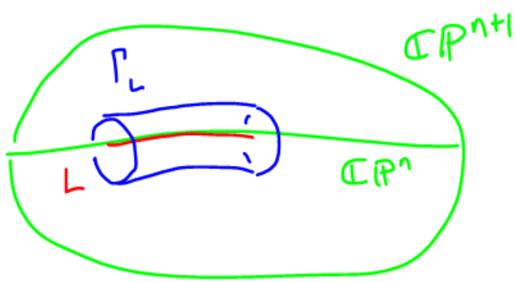
$h \in QH^2$ is invertible

Multiplication by $CO^0(h)$ gives $HF^* \xrightarrow{\cong} HF^{*+2}$

- Damian's Floer theory lifted to the univeral cover

Or twisted by local coefficients pushed forward from cover

- Biran circle bundle construction



$$L \subset \mathbb{CP}^n \rightsquigarrow \Gamma_L \subset \mathbb{CP}^{n+1} \setminus \mathbb{CP}^n = \mathbb{C}^{n+1}$$

displaceable so $HF^*(\Gamma_L, \Gamma_L) = 0$

Drawbacks

(1) Standard Floer theory is restricted to $\mathbb{Z}/2$ coefficients unless L is orientable and spin.

can relax to relatively spin ie $w_2(L) = b|_L$ for some $b \in H^2(X;\mathbb{Z}/2)$

For $\mathbb{RP}^n$ this requires $n \equiv 3 \mod 4$ (or $n=1$)

(n odd for relatively spin)

(2) It's unsatisfying to pass to the auxiliary space Γ_L. Everything should be visible to Floer theory on L itself.

New results $L \subset \mathbb{C}P^n$ monotone, $N_L = n+1$

Thm (Konstantinov–S, '18): For any n

$\tilde{L}$ is homeomorphic to S^n and $\pi_1(L) \cong \mathbb{Z}/2$

Proof avoids the circle bundle Γ_L

Corollary 1: L is homotopy equivalent to $\mathbb{R}P^n$

Corollary 2: If $n \leq 3$ then L is diffeomorphic to $\mathbb{R}P^n$

Application to nearby Lagrangian conjecture

Have $\mathbb{CP}^n = D^*\mathbb{RP}^n \cup Q^{n-1}$

weinstein n'hood of $\mathbb{RP}^n$

quadric hypersurface

If $L \subset T^*\mathbb{RP}^n$ exact Maslov zero Lagrangian

then can implant in $\mathbb{CP}^n$ as a

monotone Lagrangian with $N_L = n+1$

Thm $\Longrightarrow$ $L \simeq \mathbb{RP}^n$

[Known from Abouzaid, Kragh by different methods]

Main ingredients in Thm (Assume $n \geq 3$ from now on)

- Oh spectral sequence
- $\mathcal{CO}^0 : QH^* \longrightarrow HF^*$
- Floer theory lifted to covers
- Zapolsky's canonical pearl complex for HF^*_{Zap}

Allows us to weaken "orientable and relatively spin" to relatively pin

$\exists\, b \in H^2(X; \mathbb{Z}/2)$ with $b|_L = w_2(L)$ or $w_2(L) + w_1(L)^2$

Lifted Floer theory (Damian)

R coeffient ring

$L' \xrightarrow{\pi} L$ cover

constant sheaf on L

constant sheaf on L'

$$HF^*(L, L'; R) := HF^*((L, \underline{R}), (L, \pi_* \underline{R}))$$

Properties

- Module over $HF^*(L, L; R) = HF^*((L, \underline{R}), (L, \underline{R}))$

  Hence also over QH^* via $\mathcal{CO}^0$

- Oh spectral sequence has $E_1 = H^*_c(L'; R) = H^*_m(L; \pi_* \underline{R})$

  compactly supported

- Needs $N_L \geq 3$ in general to ensure $d^2 = 0$ — ok for us

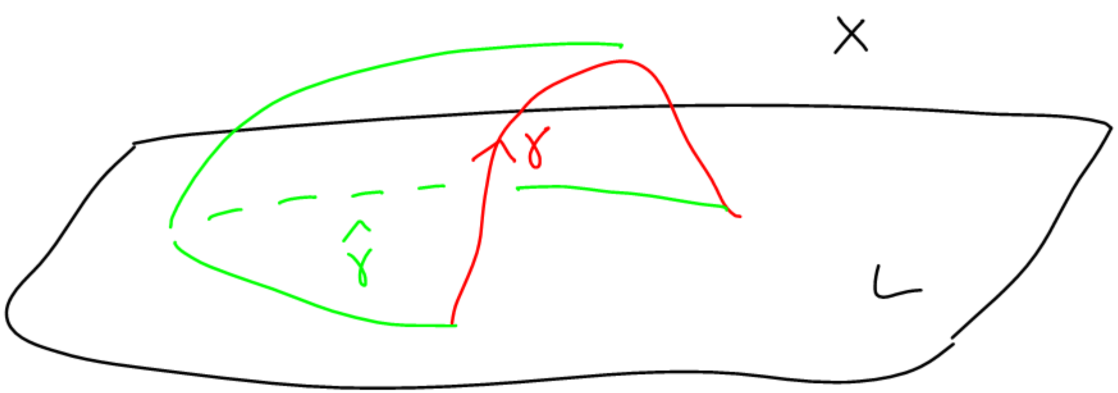
Zapolsky's setup

Usually $CF^*(L,L)$ is generated by pairs $(\gamma, \hat{\gamma})$ where

- γ is a contractible Hamiltonian chord $L \rightsquigarrow L$
- $\hat{\gamma} \in \{$htpy classes of cappings of $\gamma\} / \sim$

with $\hat{\gamma}_1 \sim \hat{\gamma}_2 \iff \hat{\gamma}_1 \# -\hat{\gamma}_2$ has Maslov index 0

Action of Novikov variable is by changing the capping



To go beyond $\mathbb{Z}/2$-coefficients need to compare orientations for $\bar{\partial}$-operators on $\hat{\gamma}_1, \hat{\gamma}_2$ whenever $\hat{\gamma}_1 \sim \hat{\gamma}_2$

Zapolsky's setup

CF^*_{Zap} is generated by $(\gamma, \hat{\gamma})$ where $\sim$ is replaced by a finer relation $\sim_{Zap}$ st all necessary orientations can be compared. Requires topological condition

$\Uparrow$

relatively pin

Complex may be much larger than usual CF^*

There's a quotient procedure for making it smaller

Properties

- Have QH^*_{Zap} and $CO^0 : QH^*_{Zap} \to HF^*_{Zap}$
- Have an spectral sequence. E_1 more complicated but contains $H^*(L)$ comes from constant capping
- HF^*_{Zap} is NOT in general a module over Novikov ring

Zapolsky's setup – Example of $\mathbb{RP}^n / \mathbb{Z}$

Let $\mathcal{L} \to \mathbb{RP}^n$ be the orientation local system $/\mathbb{Z}$

Monodromy described by $w_1(L)$

Note $H^*(\mathbb{RP}^n; \mathcal{L}) \cong H_{n-*}(\mathbb{RP}^n)$ by Poincaré duality

Oh spectral sequence has

$$E_1 = \cdots \to H^*(\mathbb{RP}^n; \mathcal{L}) \xrightarrow{-h} H^*(\mathbb{RP}^n) \xrightarrow{-h} H^*(\mathbb{RP}^n; \mathcal{L}) \xrightarrow{-h} H^*(\mathbb{RP}^n) \to \cdots$$

Know $CO^0(2h) = 0$ since $PD(\text{Quadric})|_{\mathbb{RP}^n} = 0$

But h is invertible so HF^*_{Zap} is 2-torsion and 2-periodic $\quad HF^*_{Zap} \cong HF^{*+2}_{Zap}$

$\therefore$ Differential $H^n(\mathbb{RP}^n; \mathcal{L}) \to H^0(\mathbb{RP}^n)$ is $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

Obtain $HF^i_{Zap} \cong \begin{cases} \mathbb{Z}/2 & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

If we hadn't twisted by $\mathcal{L}$, couldn't achieve 2-torsion + 2-periodic

Sketch proof of main thm

$\tilde{L} \cong S^n$

$\pi_1(L) \cong \mathbb{Z}/2$

appears already in Biran–Cornea

- Consider ordinary HF^* over $\mathbb{Z}/2$ and show $\mathcal{CO}^0 : QH^2 \longrightarrow HF^2$ is $\cong$

  Deduce $H^2(\mathbb{CP}^n; \mathbb{Z}/2) \to H^2(L; \mathbb{Z}/2)$ is $\cong$ so L relatively pin

  Also get $H^*(L; \mathbb{Z}/2) \cong H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ additively

- Now consider $HF^*_{\mathbb{Z}_{4p}}(L, L; \mathbb{Z})$. Oh spectral sequence $\Rightarrow$ it contains

  $H^*(L; \mathbb{Z})$ for $* = 1, \dots, n-1$. } "intermediate cohomology"

- Auroux–Kontsevich–Seidel : $\mathcal{CO}^0(2c_1(X)) = (\text{count of index 2 discs on } L) \cdot 1_L$

  For us $\mathcal{CO}^0(2(n+1)h) = 0$

- Invertibility of $h \Rightarrow$ intermediate cohomology of L is

  $2(n+1)$-torsion and 2-periodic

- H^1 is always torsion-free, so $H^1(L; \mathbb{Z}) = 0$

Sketch proof of main thm — continued

- Combining $H^1(L)=0$ and $N_L = n+1$ with LES of pair $(\mathbb{CP}^n, L)$ obtain $2h|_L = 0$ ⟵ This was automatic for $\mathbb{RP}^n$
- Deduce $CO^0(2h) = 0$ so $HF^*_{\mathbb{Z}/4}(L,L)$ is 2-torsion and 2-periodic
- For any cover $L' \to L$ get $HF^*_{\mathbb{Z}/4}(L,L')$ ——— " ———
- Oh spectral sequence $\Rightarrow HF^*_{\mathbb{Z}/4}(L,L')$ contains intermediate cohomology of L' (with cpct support)

Upshot

For any cover $L' \to L$ the groups $H^*_c(L';\mathbb{Z})$ $1 \leq * \leq n$ are 2-torsion and 2-periodic

Sketch proof of main thm – finishing off

For any cover $L' \to L$ the groups $H^*_c(L';\mathbb{Z})$ $1 \leq * \leq n$ are 2-torsion and 2-periodic

- Apply to cyclic covers $\rightsquigarrow$ every elt of $\pi_1(L)$ has order 2
  So $\pi_1(L)$ is abelian, hence $\cong H_1(L)$
  - $\pi_1(L)$ also finitely generated and 2-torsion so finite $\cong (\mathbb{Z}/2)^k$
  - Since $H^1(L;\mathbb{Z}/2) \cong H^1(\mathbb{RP}^n;\mathbb{Z}/2)$, get $k=1$ $\quad \therefore \pi_1(L) = \mathbb{Z}/2$
- Apply to $L' = \tilde{L}$. Know $H^1 = 0$, H^2 torsion-free so $\tilde{L}$ is $\mathbb{Z}$HS
  Homology Whitehead thm + Poincaré conj $\Rightarrow$ $\tilde{L}$ homeo S^n $\square$

Future directions

- What about homeomorphism/diffeomorphism type of L?
- Classification up to Ham isotopy?
- Other applications of Zapolsky's setup to non-orientable Lagrangians?

Merci pour votre
attention